2023-2024 理论经济学博士学位课程《高级计量经济学》期中考试参考答案

一、

1. $\Pr(A) = \frac{1}{6}$，$\Pr(B) = \frac{1}{6}$，$\Pr(A \cap B) = \frac{1}{36} = \Pr(A) \cdot \Pr(B)$，因此 A 和 B 独立
2. $\Pr(A \mid C) = \frac{1}{6}$，$\Pr(B \mid C) = \frac{1}{6}$，$\Pr(A \cap B \mid C) = \frac{1}{6} \neq \Pr(A \mid C) \cdot \Pr(B \mid C)$，因此 A 和 B 关于 C 条件非独立

二、

$$\mathrm{E}[a + bX] = a + b\mathrm{E}[X]$$

$$\begin{aligned}
\mathrm{Var}[a + bX] &= \mathrm{E}\left[(a + bX - a - b\mathrm{E}[X])^2\right] \\
&= \mathrm{E}\left[b^2(X - \mathrm{E}[X])^2\right] \\
&= b^2\mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\
&= b^2\mathrm{Var}[X]
\end{aligned}$$

三、

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty x\,\tfrac{1}{\lambda}\exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= \left[x\,\tfrac{1}{\lambda}(-\lambda)\exp\left(-\tfrac{x}{\lambda}\right)\right]_0^\infty - \int_0^\infty -\exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= 0 + \int_0^\infty \exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= \left[(-\lambda)\exp\left(-\tfrac{x}{\lambda}\right)\right]_0^\infty = 0 - (-\lambda) = \lambda
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[X^2] &= \int_0^\infty x^2\,\tfrac{1}{\lambda}\exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= \left[x^2\,\tfrac{1}{\lambda}(-\lambda)\exp\left(-\tfrac{x}{\lambda}\right)\right]_0^\infty - \int_0^\infty -2x\exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= 0 + 2\lambda\int_0^\infty x\,\tfrac{1}{\lambda}\exp\left(-\tfrac{x}{\lambda}\right)dx \\
&= 2\lambda\lambda = 2\lambda^2
\end{aligned}$$

$$\Rightarrow \quad \mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

四、

$$
\begin{aligned}
\mathrm{E}[Y] &= \int y f_Y(y) dy = \int y \Big(\int f_{X,Y}(x,y) dx \Big) dy \\
&= \int\int y f_{X,Y}(x,y) dx dy = \int\int y f_{Y|X}(y \mid x) f_X(x) dx dy \\
&= \int \Big(\int y f_{Y|X}(y \mid x) dy \Big) f_X(x) dx = \int \mathrm{E}[Y \mid X = x] f_X(x) dx = \mathrm{E}\big[\mathrm{E}[Y \mid X]\big]
\end{aligned}
$$

五、

1.

$$
\begin{aligned}
&\mathrm{E}\big[(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])^\top\big] \\
&= \mathrm{E}\left[\begin{pmatrix} X_1 - \mathrm{E}[X_1] \\ X_2 - \mathrm{E}[X_2] \\ \vdots \\ X_n - \mathrm{E}[X_n] \end{pmatrix} \big(X_1 - \mathrm{E}[X_1], X_2 - \mathrm{E}[X_2], \cdots, X_n - \mathrm{E}[X_n]\big)\right] \\
&= \mathrm{E}\Big[\big((X_i - \mathrm{E}[X_i])(X_j - \mathrm{E}[X_j])\big)_{ij}\Big] = \boldsymbol{\Sigma}
\end{aligned}
$$

2. 对任意的 $\boldsymbol{w} \in \mathbb{R}^n$，$\boldsymbol{w}^\top(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}]) = z \in \mathbb{R}$，此时，

$$
\begin{aligned}
\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} &= \boldsymbol{w}^\top \mathrm{E}\big[(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])^\top\big] \boldsymbol{w} \\
&= \mathrm{E}\big[\boldsymbol{w}^\top(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])^\top \boldsymbol{w}\big] \\
&= \mathrm{E}\Big[\big(\boldsymbol{w}^\top(\boldsymbol{X} - \mathrm{E}[\boldsymbol{X}])\big)^2\Big] = \mathrm{E}[z^2] \geq 0
\end{aligned}
$$

因此 $\boldsymbol{\Sigma}$ 是半正定矩阵。

六、

1. 参数 θ 的估计量 $\hat{\theta}$ 的估计偏差是 $\mathrm{bias}[\hat{\theta}] = \mathrm{E}[\hat{\theta}] - \theta$
2. 当 $\mathrm{bias}[\hat{\theta}] = 0$ 时，称 $\hat{\theta}$ 为 θ 的非偏估计量

七、令 $\boldsymbol{X} = (\boldsymbol{\iota} \quad \boldsymbol{D})$，则根据 OLS 估计量的表达式 $\boldsymbol{\beta} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}$，

$$
\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y} \\
&= \left(\begin{pmatrix} \boldsymbol{\iota}^\top \\ \boldsymbol{D}^\top \end{pmatrix} (\boldsymbol{\iota} \quad \boldsymbol{D}) \right)^{-1} \begin{pmatrix} \boldsymbol{\iota}^\top \\ \boldsymbol{D}^\top \end{pmatrix} \boldsymbol{y} = \begin{pmatrix} n & \sum D_i \\ \sum D_i & \sum D_i \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum_{D_i=1} y_i \end{pmatrix} \\
&= \frac{1}{mn - m^2} \begin{pmatrix} m & -m \\ -m & n \end{pmatrix} \begin{pmatrix} n\bar{y} \\ m\bar{y}_1 \end{pmatrix} = \frac{1}{mn - m^2} \begin{pmatrix} mn\bar{y} - m^2\bar{y}_1 \\ -mn\bar{y} + mn\bar{y}_1 \end{pmatrix}
\end{aligned}
$$

因 $\sum y_i = \sum_{D_1=1} y_i + \sum_{D_i=0} y_i$，可知 $n\bar{y} = (n-m)\bar{y}_0 + m\bar{y}_1$，则

$$\alpha = \frac{mn\bar{y} - m^2\bar{y}_1}{mn - m^2} = \frac{m(n-m)\bar{y}_0 + m^2\bar{y}_1 - m^2\bar{y}_1}{mn - m^2} = \bar{y}_0$$

$$\beta = \frac{-mn\bar{y} + mn\bar{y}_1}{mn - m^2} = \frac{-m(n-m)\bar{y}_0 - m^2\bar{y}_1 + mn\bar{y}_1}{mn - m^2}$$

$$= \frac{(mn - m^2)(\bar{y}_1 - \bar{y}_0)}{mn - m^2} = \bar{y}_1 - \bar{y}_0$$